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On One Dimensional Submodules and Hyperoctahedral Groups

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This note is a kind of postscript to [*J. Algebra* (1980), in press], where the irreducible representations of the hyperoctahedral groups over a field are constructed. The purpose of this note is to determine the one dimensional submodules of the Young ideals involved in the construction. We assume familiarity with [*J. Algebra* (1980), in press], and the notation is not redefined here. The method used may be applied in the case of the symmetric groups to yield the well-known results of G. D. James on the trivial submodules of Specht modules.

Let (λ, μ) be a pair of partitions. The *conjugate* of (λ, μ) is defined to be the pair (μ', λ'), xhere μ', λ' are the conjugates of μ and λ, respectively. If x is a (λ, μ) tableau consisting of parts x_λ, x_μ, the conjugate tableau is $x' = (x'_{\mu'}, x'_{\lambda'})$ obtained by conjugating each part and changing the order. Thus $Rx = Cx'$ and $Cx = Rx'$.

1. THEOREM. *For $\xi \in KW_n$, $\xi \in KW_n \varepsilon(Cx)\iota(Rx)$ if and only if*

(i) $\xi(1 - (a, b)(-a, -b)) = 0$ *for all pairs a, b in the same row of x.*

(ii) $\xi(1 - \alpha_a(a, -a)) = 0$ *for all a, where $\alpha_a = 1$ if a is in x_λ and $\alpha_a = 1$ if a is in x_μ.*

Let (s, t, k), $(s + 1, t, k) \in D_{\lambda,\mu}$. Let D be the set of entries of x in row s, to the right of (s, t, k), including $x(s, t, k)$. Let E be the set of entries in x in row $s + 1$ to the left of $(s + 1, t, k)$ including $x(s + 1, t, k)$.

Let $F = E \cup D$. Let S_D, S_E, S_F be the subgroups of W_n generated by the positive permutations on D, E, F, respectively, and let Y be a system of left coset representatives of $S_D S_E$ consisting of transpositions.

(iii) $\xi\iota(Y) = 0$ *for each (s, t, k), $(s + 1, t, k) \in D_{\lambda,\mu}$.*

Note. Condition (i) and those cases of (ii) for which $\alpha_a = -1$ guarantee that $\xi \in KW_n\iota(Rx)$. Condition (ii) may be interpreted saying that the Young

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ideal is an intersection of kernels of homomorphisms with co-domain $KW_n\imath(Rx)$ (cf. [3, Theorem 9.3]).

Proof. First, $\xi \in KW_n\varepsilon(Cx)\imath(Rx)$ if and only if ξ belongs to the left annihilator ideal of the right annihilator of the element $\varepsilon(Cx)\imath(Rx)$ because KW_n is a Frobenius algebra. To find this right annihilator, we use the unique anti-isomorphism of KW_n which maps g to its inverse. We denote the image of ξ under this map by $\bar{\xi}$. We also use the ring isomorphism $T\colon KW_n \to KW_n$ defined by

$$T(\Sigma a_w w) = \Sigma a_w \varepsilon(w) w \qquad (w \in W_n).$$

Applying these two maps, we see that $\varepsilon(Cx)\imath(Rx)\psi = 0$ if and only if $T(\bar{\psi})\,\varepsilon(Cx')\imath(Rx') = 0$, if and only if ψ is a linear combination of $T(\bar{v})\eta$; where $\eta \in KW_n$ and v is one of the generators of the left annihilator ideal of $\varepsilon(Cx')\imath(Rx')$ described in [1, Sect. 3]. The elements used in (i–iii) of this theorem are precisely the $T(\bar{v})$ for these v. This completes the proof.

2. Corollary. *$Y^{\lambda,\mu}$ contains the trivial submodule $K\imath(W_n)$ if and only if $\mu = \phi$ or the characteristic of K is 2, and if*

$$\gamma_s \equiv -1 \bmod(p^{l(\gamma_{s+1})}) \tag{3}$$

for $\gamma_s = \lambda_s$ $(1 \leqslant s < \lambda_1)$ and for $\gamma_s = \mu_s$ $(1 \leqslant s < \mu_1')$, where $l(\gamma_{s+1}) = h$ if $p^{h-1} \leqslant \gamma_{s+1} < p^h$.

Proof. A necessary and sufficient condition for $Y^{\lambda,\mu}$ to contain a submodule isomorphic to K is that $KW_n\varepsilon(x)\imath(Rx)$ contains $\imath(W_n)$. Condition (i) of the theorem is clear. Condition (ii) is satisfied if and only if $\mu = \phi$ or the characteristic of K is two. The number of terms in the element $\imath(Y)$ of condition (iii) is the binomial coefficient

$$\binom{\gamma_{s+1}}{t}, \qquad (1 \leqslant t \leqslant \gamma_{s+1}).$$

These coefficients are all zero mod p if and only if (3) is satisfied [2].

3. Corollary. *Suppose the ground field has characteristic not 2.*

(i) *$Y^{\lambda,\mu}$ contains a submodule isomorphic to $KW_n\xi(W_n)$ if and only if $\lambda = \phi$ and for $s = 1, 2,\ldots$*

$$\mu \equiv -1 \bmod(p^{l(\mu_{s+1})})$$

(ii) *$Y^{(\lambda,\mu)}$ contains a submodule isomorphic to $KW_n\eta(W_n)$ if and only if $\lambda = \phi$ and $\mu = (1^n)$.*

The proof of 3 is left as an exercise. Part (iii) is in agreement with [1, (2.6), (ii)].

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